ON THE PLANE PROBLEM FOR A WEDGE

(O PLOSKOI ZADACHE DLIA KLINA)

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Some solutions of this problem are known for loading of the wedge on its sides or at its apex; for example, the cases shown in Fig. 1. However, each of these solutions is nonunique because the boundary conditions are specified only on the boundaries of the wedge, which are assumed to be infinite. The behavior of the stresses at infinity is not stipulated beforehand.

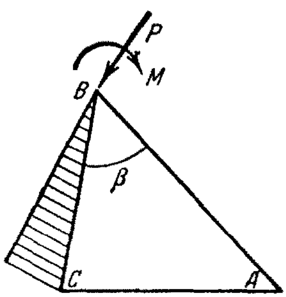


Fig. 1.

Indeed, examine the upper portion of a wedge (Fig. 2) which is cut off by the segment *AC*, i.e. the triangle *ABC*, and assume that this section is subjected along *AC* to an arbitrary self-equilibrating load, and that the boundaries *BA* and *BC* remain free. This gives rise in the triangle to a state of stress that depends on the character of the load applied along the boundary *AC*. Superpose the state of stress obtained in this manner on the state that exists in one of the problems indicated above. The boundary conditions are then retained, but the state of stress varies depending on the loading along the face *AC*. If the load is assumed to be arbitrary, except for the restriction that its resultant moment and load vector be equal to zero, then we obtain an infinite set of solutions for a given problem with given boundary conditions on *BA* and *BC* – as was to be proved. Hence it follows that for the complete solution of the wedge problem it is of interest to solve the problem of a triangle (Fig. 2) which has an arbitrary self-equilibrating load applied on one face *AC* and has the remaining two faces *BA* and *BC* free.

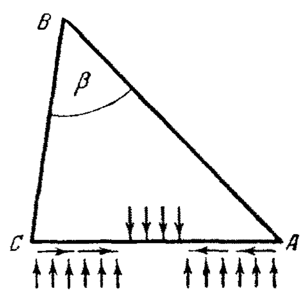


Fig. 2.

1. In the present work the variational method of Castigliano is applied to the problem indicated. We represent the unknown stress tensor as the sum of two tensors

$$X_x = X_x^\circ + \Sigma A_m X_x^{(m)}, \qquad Y_y = Y_y^\circ + \Sigma A_m Y_y^{(m)}$$
$$X_y = X_y^\circ + \Sigma A_m X_y^{(m)} \tag{1}$$

Here the tensor whose components are denoted by the index 0 is the basic tensor, which satisfies the differential equations of equilibrium and the boundary conditions of the problem (Fig. 2). The second terms in Formulas (1) comprise the correction tensor. This tensor likewise satisfies the equations of equilibrium but leaves the boundaries of the region *ABC* free of stress. It is required that each of the tensors being summed, with components $X_x^{(m)}$, $Y_y^{(m)}$, $X_y^{(m)}$, satisfy these two conditions. Then the presence of the arbitrary constants A_m allows one to carry out the variation of the general tensor (1) that is necessary to form the variational equation of Castigliano

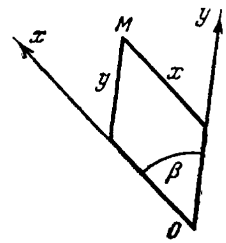


Fig. 3*a*.

$$\delta \int\limits_{(\sigma)} W\, d\sigma = \int\limits_{(s)} (\delta X_\nu u + \delta Y_\nu v)\, ds \tag{2}$$

In the present problem the surface forces X_ν, Y_ν are given and therefore are not varied. Thus Equation (2) simplifies and leads to the theorem of minimum elastic energy

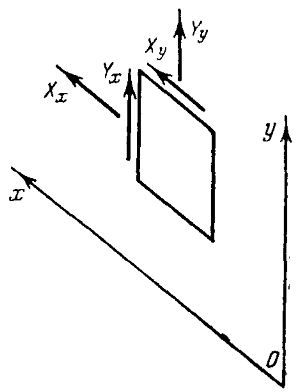


Fig. 3*b*.

$$\delta V = \delta \int\limits_{(\sigma)} W\, d\sigma = 0 \tag{3}$$

As a result of substituting Expression (1) into the function *W* and carrying out the integration, we obtain for the function *V* on the left side of Equation (3) a quadratic function in the arbitrary constants A_m. To determine these we arrive at a system of linear equations

$$\partial V / \partial A_m = 0 \qquad (m = 1, 2, \ldots) \tag{4}$$

2. In the present problem it is convenient to use a skewed (contravariant) coordinate system *x*, *y* (Fig. 3*a*) instead of rectangular coordinates. As the coordinate axes we take the sides of the wedge *BA* and *BC*, with the angle β between them. Then the "skewed" contravariant components

of the stress tensor X_x, Y_y and X_y shown in Fib. 3*b* may be designated in the same way as in a rectangular coordinate system but with this difference: the subscript does not denote the external normal to an area element that is parallel to one of the coordinate axes but rather denotes the direction of the other axis such that, for example, X_x denotes the projection (oblique angled) onto the x-axis of the entire stress acting on an area element with the "pseudo normal" x (i.e. onto an area element parallel to the y-axis).

3. It is easy to show that in the skewed [oblique] coordinate system that has been adopted the equilibrium equations have the usual form

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} = 0, \qquad \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} = 0 \tag{5}$$

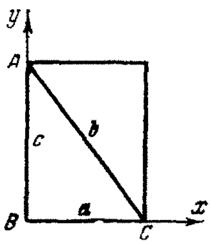


Fig. 4.

Likewise, the shear components X_y and Y_x that we have introduced satisfy the conjugate relation $Y_x = X_y$ (these shear components differ from the usual shear stresses).

The basic and correction stress tensors must satisfy the differential equations of equilibrium. This will be the case if each of them is obtained from an Airy stress function $\varphi(x, y)$ in accordance with the formulas

$$\begin{aligned} X_x^{(0)} &= \frac{\partial^2 \varphi_0}{\partial y^2}, & Y_y^{(0)} &= \frac{\partial^2 \varphi_0}{\partial x^2}, & X_y^{(0)} &= -\frac{\partial^2 \varphi_0}{\partial x \partial y} \\ X_x^{(k)} &= \frac{\partial^2 \varphi_k}{\partial y^2}, & Y_y^{(k)} &= \frac{\partial^2 \varphi_k}{\partial x^2}, & X_y^{(k)} &= -\frac{\partial^2 \varphi_k}{\partial x \partial y} \end{aligned} \tag{6}$$

4. We use the following considerations to construct the basic stress tensor. In the case of a rectangle (Fig. 4) with sides $x = 0$, $y = 0$, $x = a$, $y = c$ the correction stress tensor may be obtained in a rectangular coordinate system by means of the stress function

$$\varphi_k = \sum_m \sum_n A_{mn} P_m(x) P_n(y) \tag{7}$$

Here the functions $P_m(x)$ and $P_n(y)$ are so-called cosine-binomials

$$P_m = \cos\frac{m\pi x}{a} - \cos\frac{(m+2)\pi x}{a}, \qquad P_n = \cos\frac{n\pi y}{c} - \cos\frac{(n+2)\pi y}{c} \tag{8}$$

The cosine-binomials satisfy the following boundary conditions

$$\begin{aligned} P_m(0) &= 0, & P_m'(0) &= 0, & P_n(0) &= 0, & P_n'(0) &= 0 \\ P_m(a) &= 0, & P_m'(a) &= 0, & P_n(c) &= 0, & P_n'(c) &= 0 \end{aligned} \tag{9}$$

Systems of these functions are complete and closed. If we express the components of the stress tensor in formulas of the type (6) by means of the stress tensor (7), then we note, that as a consequence of the boundary properties of the cosine-binomials (9), the entire contour of the rectangle will be free of stresses. By this means we obtain the correction tensor for the rectangle. If a right triangle is now cut out of the rectangle by a diagonal (or by another straight line), its legs will be free of stresses but there will be normal and tangential stresses that form a self-equilibrating load on the hypotenuse. The completeness of the function system (8) and the arbitrariness of the coefficients A_{mn} allow one to realize an extremely wide wide class of functions by this means.

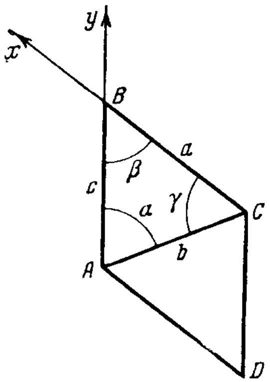
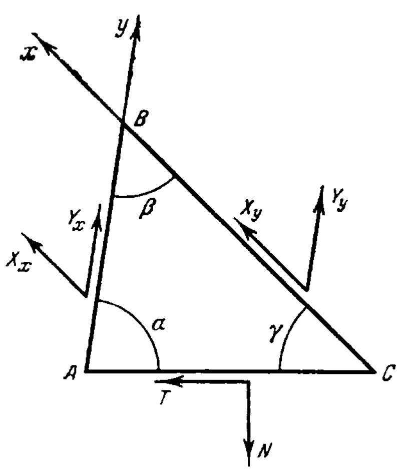


Fig. 5.

All that has been stated can be applied to the present problem of a scalene triangle. To do this, we extend the triangle (Fig. 5) to the parallelogram *ABCD* and use the stress function (7) in the skewed coordinate system that has been adopted. The contour of the parallelogram remains free of stresses in this process, while a self-equilibrated load of sufficient arbitrariness occurs on its diagonal *AC*. This means that the basic



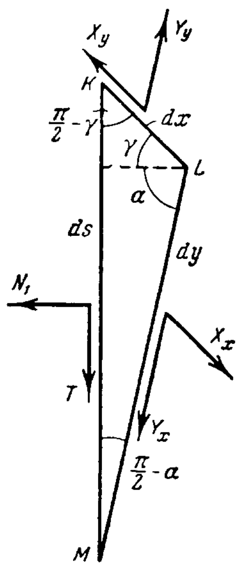


Fig. 6.

tensor for the triangle is obtained as the correction tensor for the parallelogram.

In the application of the solution obtained, it is convenient to express the equilibrated loads on the section AC in terms of a normal stress N and a shear stress T. We obtain the expressions for N and T in terms of the skewed components from the equations of equilibrium of an infinitesimal triangular element (Fig. 6)

$$\begin{gathered} N = \frac{1}{\sin\beta}(X_x \sin^2\gamma + Y_y \sin^2\alpha + 2X_y \sin\alpha \sin\gamma) \\ T = \frac{1}{\sin\beta}[-X_x \sin\gamma\cos\gamma + Y_y \sin\alpha\cos\alpha + X_y(\sin\gamma\cos\alpha - \cos\gamma\sin\alpha)] \end{gathered} \tag{10}$$

In the case of rectangular coordinate system ($\beta = \pi/2$, $\gamma = \pi/2 - \alpha$) these formulas coincide with the formulas for the "stresses on an oblique element" that are used in Strength of Materials. We write the equation for the section AC in the form (Fig. 5):

$$x/a + y/c = 1 \tag{11}$$

For the orientation of the triangle in Fig. 5 the segments a and c are negative. On the straight line AC the variable y must be considered as a function of x

$$y = c(1 - x/a)$$

Corresponding to this, the arguments of the function $P_n(y)$ and its derivatives are transformed to

$$\frac{n\pi y}{c} = n\pi\left(1 - \frac{x}{a}\right) = n\pi - \frac{n\pi x}{a}$$

Summarizing the results, we have

$$\begin{array}{ll} \text{For } n \text{ even} & \text{For } n \text{ odd} \\ P_n(y) = P_n(x), & P_n(y) = -P_n(x) \\ P_n'(y) = -\dfrac{a}{c}P'_n(x), & P_n'(y) = \dfrac{a}{c}P_n'(x) \\ P_n''(y) = \dfrac{a^2}{c^2}P_n''(x), & P_n''(y) = -\dfrac{a^2}{c^2}P_n''(x) \end{array} \tag{12}$$

From this summary it follows that on the straight line AC the function $P_n(y)$, as considered on the segment c, passes over to the function $P_n(x)$, as considered on the segment a. Hence the stresses N and T on the section AC are expressible as a function of a single (skewed) coordinate x. To find these expressions it is necessary to obtain the components X_x, Y_y and X_y from the stress function (7)

$$X_x^{(0)} = \sum\sum A_{mn} P_m(x) P_n''(y), \qquad Y_y^{(0)} = \sum\sum A_{mn} P_m''(x) P_n(y)$$

$$X_y^{(0)} = -\sum\sum A_{mn} P_m'(x) P_n'(y) \tag{13}$$

and to transform $P_n(y)$ and its derivatives in accordance with (12). Omitting the corresponding calculations, we deduce the final expressions N and T (in obtaining these we have applied the law of sines to the triangle ABC).

We have

$$N = \frac{\sin^2 \alpha}{\sin \beta} \sum\sum (-1)^n A_{mn} [P_m(x) P_n(x)]'' \tag{14}$$

$$T = \frac{\sin^2 \alpha}{\sin \beta} \left\{ \cot \alpha \sum\sum (-1)^n A_{mn} [P_m'(x) P_n(x)]' - \cot \gamma \sum\sum (-1)^n A_{mn} [P_m(x) P_n'(x)]' \right\}$$

Primes in the expressions in the square brackets denote derivatives with respect to the variable x. By the use of Expression (14), we evaluate the resultant load and moment of the forces applied along the section AC

$$\begin{aligned}
\int_0^l N(s)\, ds &= \int_0^a N(x) \frac{dx}{\sin \alpha} = \frac{1}{\sin \alpha} \int_0^a N(x)\, dx \\
\int_0^l sN(s)\, ds &= \int_0^a \frac{x}{\sin \alpha} N(x) \frac{dx}{\sin \alpha} = \frac{1}{\sin^2 \alpha} \int_0^a xN(x)\, dx \\
\int_0^l T(s)\, ds &= \int_0^a T(x) \frac{dx}{\sin \alpha} = \frac{1}{\sin \alpha} \int_0^a T(x)\, dx
\end{aligned} \tag{15}$$

Substituting into the above from (14), we note that total derivatives stand under the integral sign. Hence, the integrations are performed without difficulty, and evaluation at the limits of integration and a consideration of the boundary properties of the cosine-binomials shows that all of the quantities (15) vanish and the force on the cross-section AC is self-equilibrating – as was established *a priori*.

If the normal and tangential stresses $N^{\circ}(x)$ and $T^{\circ}(x)$ on the section AC are given (their distribution is expressed as a function of x, as in (15)) they must be approximated by means of Formula (14). For example, this can be done by the method of minimum quadratic deviation, i.e. by requiring that the integral

$$J = \int_0^a [(N - N^{\circ})^2 + (T - T^{\circ})^2]\, dx$$

take on a minimum value. Equating to zero the derivatives with respect to the parameters A_{mn}, we obtain a system of linear equations for their determination

$$\frac{\partial J}{\partial A_{mn}} = 0 \qquad \begin{pmatrix} m = 0,\ 1,\ 2, \ldots \\ n = 0,\ 1,\ 2, \ldots \end{pmatrix}$$

By this means the construction of the basic stress tensor is completed.

5. The basic tensor was constructed with the aid of the stress function (7). In this process the sides BA and BC of the triangle turned out to be free of stress as a consequence of the boundary properties (9) of the cosine-binomials. To construct the correction tensor we affix to each term of the sum (7) an analogous factor that together with its derivative goes to zero on the straight line AC. For this purpose we introduce a new argument

$$w = x/a + y/c \tag{16}$$

Because of the equation of the straight line AC (11), $w = 1$ on AC, so, for example, the function

$$P(w) = 1 - \cos 2\pi w \tag{17}$$

satisfies the required conditions; in fact

$$P(w) = 0, \qquad \frac{dP}{dw} = P'(w) = 2\pi \sin 2\pi w = 0 \tag{18}$$

We form the correction tensor with the aid of the stress function

$$\varphi_k = P(w) \sum\sum C_{mn} P_m(x) P_n(y) \tag{19}$$

We write out the components of the stress tensor; for brevity we retain only a single summation sign; and in addition we omit the arguments of the functions $P_m = P_m(x)$, $P_n = P_n(y)$ and $P = P(w)$.

We denote the derivatives of these functions by primes, so that we have

$$X_x^{(k)} = \frac{\partial^2 \varphi_k}{\partial y^2} = \frac{1}{c^2} P'' \Sigma C_{mn} P_m P_n + 2\frac{1}{c} P' \Sigma C_{mn} P_m P_n' + P \Sigma C_{mn} P_m P_n''$$

$$Y_y^{(k)} = \frac{\partial^2 \varphi_k}{\partial x^2} = \frac{1}{a^2} P'' \Sigma C_{mn} P_m P_n + 2\frac{1}{a} P' \Sigma C_{mn} P_m' P_n + P \Sigma C_{mn} P_m'' P_n$$

$$X_y^{(k)} = -\frac{\partial^2 \varphi_k}{\partial x \partial y} = -\frac{1}{ac} P'' \Sigma C_{mn} P_m P_n - P' \left(\frac{1}{a} \Sigma C_{mn} P_m P_n' + \frac{1}{c} \Sigma C_{mn} P_m' P_n \right) - P \Sigma C_{mn} P_m' P_n' \tag{20}$$

These formulas give the correction tensor for the triangle, as can be verified immediately. The skewed components (20) vanish on the sides *BA* and *BC*. This follows from the boundary properties of the cosine-binomials (9). On the side *AC* we form the components (10). We note that Formula (20) simplifies considerably on the straight line *AC*. Here $P = 0$, $P' = 0$ according to (17) and (18). Retaining only the first terms of Formulas (20) and substituting in Formulas (10), we obtain

$$N = \frac{1}{\sin\beta}\left[\frac{\sin^2\gamma}{c^2} + \frac{\sin^2\alpha}{a^2} - \frac{2\sin\alpha\sin\gamma}{ac}\right] P'' \sum C_{mn} P_m P_n$$

$$T = \frac{1}{\sin\beta}\left[-\frac{\sin\gamma\cos\gamma}{c^2} + \frac{\sin\alpha\cos\alpha}{a^2} - \frac{\sin\gamma\cos\alpha - \cos\gamma\sin\alpha}{ac}\right] P'' \sum C_{mn} P_m P_n \qquad (21)$$

Here the expressions in the square brackets vanish as a consequence of the law of sines and hence the stresses *N* and *T* are absent on the side *AC* for arbitrary values of the parameters C_{mn}, as was to be proved.

6. The complete stress tensor of the given problem is obtained in accordance with (1) as the sum of the correction tensor (20) and the basic tensor. The latter is formed from the stress function (7)

$$X_x^{(0)} = \sum\sum A_{mn} P_m(x) P_n''(y), \qquad Y_y^{(0)} = \sum\sum A_{mn} P_m''(x) P_n(y)$$
$$X_y^{(0)} = -\sum\sum A_{mn} P_m'(x) P_n'(y) \qquad (22)$$

The coefficients A_{mn} are determined from the conditions for the approximation of the loads on the side *AC*, as was indicated in Section 4.

In order to formulate the Castigliano variational equation, (3), it remains to construct the expression for the elastic energy *W*. If we refer the triangle being studied to a rectangular coordinate system, one of whose axes is parallel to the side *AC*, then the components *N* and *T* introduced above will enter in as components of the stress tensor (Fig. 6*a*). In addition to these, it is necessary to add the component N_1 (Fig. 6*b*). The latter is expressible in terms of the skew components X_x, Y_y, X_y from the conditions of equilibrium of the element *KLM*, in analogy to (10)

$$N_1 = \frac{1}{\sin\beta}\left(X_x\cos^2\gamma + Y_y\cos^2\alpha - 2X_y\cos\alpha\cos\gamma\right) \qquad (23)$$

Adding this relation to (10), we obtain a complete system of formulas for the transformation of the stress tensor. The expression for the elastic energy is

$$W = \frac{1}{2}\left(N\varepsilon_N + N_1 e_{N_1} + T\gamma\right) = \frac{1}{2E}\left[(N + N_1)^2 - 2(1+\nu) NN_1 + 2(1+\nu) T\right] \quad (24)$$

Applying Formulas (23) and (10), we have, therefore,

$$W = \frac{1}{2E\sin^2\beta}\{(X_x + Y_y)^2 - 2(1+\nu)\sin^2\beta X_x Y_y + 4\cos\beta\, X_y(X_x + Y_y) + \\ + 2[1 + \nu + (1-\nu)\cos^2\beta]\, X_y^2\} \tag{25}$$

In obtaining these results, we took into account that

$$\alpha + \gamma = \pi - \beta, \qquad \sin(\alpha+\gamma) = \sin\beta, \qquad \cos(\alpha+\gamma) = -\cos\beta, \qquad X_y = Y_x$$

The complete elastic energy of the triangle *ABC* is

$$V = \int_{(S)} W\, dS \qquad (dS = dx\, dy \sin\beta) \tag{26}$$

Inserting this expression into (26), we obtain

$$V = \frac{1}{2E\sin\beta}\iint_{(S)}\{(X_x + Y_y)^2 - KX_xY_y + LX_y(X_x + Y_y) + MX_y^2\}\, dxdy \tag{27}$$

$$K = 2(1+\nu)\sin^2\beta, \qquad L = 4\cos\beta, \qquad M = 2[1 + \nu + (1-\nu)\cos^2\beta] \tag{28}$$

It is necessary to insert the components of the complete stress tensor into the integrand, using the Formulas (22) and (20)

$$X_x = X_x^{(0)} + X_x^{(k)}, \qquad Y_y = Y_y^{(0)} + Y_y^{(k)}, \qquad X_y = X_y^{(0)} + X_y^{(k)} \tag{29}$$

After the integration is carried out, the energy is obtained as a quadratic function of the unknown parameters C_{mn}. By the usual means, we obtain for them the system

$$\partial V / \partial C_{mn} = 0 \tag{30}$$

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